Principles of Spherical Trigonometry
Drawn from the Method of the Maxima and Minima

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Since one knows that the arcs of great circles, drawn on the surface of a sphere, represent the shortest path from one point to another, a spherical triangle could be defined thus: given three points on the surface of a sphere, let a spherical triangle be the space enclosed between these three points. Thus, since the sides of a spherical triangle are the shortest lines which can be drawn from one angle to another, the method of maxima and minima could be used to determine the sides of a spherical triangle and thence could be found the relations which subsist between the angles and sides, which is exactly the content of spherical Trigonometry. For the relationship among the three points, and the six objects, will always be such that, knowing any three of them, the other three can be determined.

This, therefore, is a property which the spherical triangles have in common with plane triangles, which are the subject of elementary Trigonometry. For, just as a plane triangle is the space enclosed between three points marked on a plane, when the shortest path is drawn from one to another, which is, on the plane, a straight line, so a spherical triangle is the space enclosed between three points marked on the surface of a sphere, when these three points are joined by the shortest lines that can be drawn on the same surface. Now, clearly, a spherical triangle changes into a planar triangle, when the radius of a sphere becomes infinitely large, so that a planar surface can be regarded as the surface of an infinitely large sphere.

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No doubt, the objection will be raised, that it goes against the rules of the method, to want to use the calculus of the maxima and minima to establish the foundations of spherical Trigonometry; it seems useless, moreover, to derive these again from other principles, since those which have been used until now, are founded on elementary Geometry, whose rigor serves as a rule for all the other parts of Mathematics. But I remark, first, that the method of maxima and minima thereby acquires somewhat of a new glow, when I shall make seen that by itself it leads to the resolution of spherical triangles; moreover, it is always useful to arrive by different routes at the same truth, since our spirits will not fail thereby to arrive at new explanations.

But I also argue that the method of maxima and minima is much more general than the ordinary method. For the latter is limited to triangles formed on plane or spherical surfaces, while the former extends to any surface whatever. Thus, if one asks the nature of triangles formed on spheroidal or conic surfaces, whose sides are the shortest lines that can be drawn from one angle to the other, the ordinary method would not be suitable to such research; it would be absolutely necessary to resort to the method of maxima and minima, without which one would not even be in a state to know the shortest lines, which form the sides of these triangles.

From this, it is understood that this research could well have great importance; for the surface of the Earth is not spherical, but spheroidal; a triangle formed on the surface of the Earth belongs to the sort of which I was just speaking. To see this, one only need imagine three points on the surface of the Earth which are joined by the shortest path which leads from one to the other, or formed by a cord stretched from one to the other; for it is thus that those triangles must be represented, which are used in the operations for the measure of the Earth. It is true that such triangles are ordinarily regarded as plane and rectilinear, and it is quite accurate, when one calculates on the foot of spherical triangles. But if one arrives at making these triangles much larger, and one wants to calculate with the greatest possible precision, one is without doubt reduced to investigate the true nature of such triangles, which cannot be known without recourse to the method of the maxima and minima.

Thus, having seen the importance of this method in the subject about which it concerns, it will do no harm to apply this method to the resolution of spherical triangles, since on one side, this inquiry will serve as a basis and a model for the resolution of triangles formed on a spheroidal surface; on the other hand it will furnish us with considerable explanations as much for spherical Trigonometry itself as for the method of the maxima and minima,
from which one will know more and better its extent and great usage. For, since one has shown that most mechanical and physical problems are resolved quite quickly by the means of this method, it will only be very agreeable to see that the same method brings such a great help to the resolution of problems in pure Geometry.

To begin this inquiry in such a manner that it applies equally to the sphere and to any spheroid whatever, I first consider two opposite points on the sphere as its poles, and the great circle equally distant from them will represent the equator, and the shortest lines drawn from a pole to each point on the equator will represent the meridians which are perpendicular to the equator. Now on the sphere, one side of a spherical triangle can be regarded as part of the equator, and if it is a right triangle, one of the sides which forms the right angle can always be supposed to be a portion of the meridian, since the two poles can be chosen at will. But it will not be the same when the surface is not spherical, but spheroidal. However, I will only speak here of spherical surfaces, reserving spheroidal surfaces for another Memoire.

**Problem 1**

1. **Given (Fig. 1) the arc** $AP$ **on the equator, and the arc** $PM$ **on the meridian** $OP$, **find, on the spherical surface, the shortest line** $AM$, **which can be drawn from point** $A$ **to point** $M$.

**Solution**
Setting the half-diameter of the sphere = 1, let

the arc of the equator \( AP = x \)      the arc of the meridian \( PM = y \).

In addition, let

the arc being looked for = \( s \);

suppose that \( AM \) is prolonged an infinitely small distance to \( m \), so that \( Mm = ds \), and that the meridian \( Omp \) is drawn through \( m \), and that the element \( Mn \) is perpendicular to \( Omp \). From this, one will have \( Pp = dx \) and \( mn = dy \), and since \( Pp \) is to \( Mn \) as the sine of the total meridian \( OP=1 \) to the sine of the arc \( OM \), or to the cosine of \( PM = y \), one will have

\[
Mn = dx \cos y
\]

and the right triangle \( Mnm \) to \( n \) gives

\[
Mm = ds = \sqrt{dy^2 + dx^2 \cos^2 y}
\]

and therefore

\[
AM = s = \int \sqrt{dy^2 + dx^2 \cos^2 y}.
\]

Thus, the problem is to find that relation between \( x \) and \( y \), so that if known values such as \( AP \) and \( PM \) are given, the value of the integral

\[
\int \sqrt{dy^2 + dx^2 \cos^2 y}
\]

becomes as small as possible. To this effect, set \( dy = p \, dx \), in order to reduce this integral to the form

\[
\int \sqrt{pp + \cos^2 y};
\]

and since I have demonstrated that when the integral formula \( \int Z \, dx \), where \( Z \) is some function of \( x \), \( y \), and \( p \), so that \( dZ = M \, dx + N \, dy + P \, dp \), must become as small or large as possible, this happens when

\[
N \, dx - dP = 0.
\]

Thus, in our case, we shall have

\[
Z = \sqrt{dy^2 + dx^2 \cos^2 y},
\]
and
\[ dZ = -\frac{\sin y \cos y}{\sqrt{pp + \cos^2 y}} dy + \frac{p}{\sqrt{pp + \cos^2 y}} dp; \]

so that
\[ M = 0, \quad N = -\frac{\sin y \cos y}{\sqrt{pp + \cos^2 y}}, \quad \text{and} \quad P = \frac{p}{\sqrt{pp + \cos^2 y}}. \]

Now, since \( M = 0 \) and therefore \( dZ = N dy + P dp \), we multiply the equation \( N \, dx - dP = 0 \) by \( p \), which, since \( dy = p \, dx \), will become \( N \, dy - p \, dP = 0 \), or \( N \, dy = p \, dP \), and, this value being substituted for \( N \, dy \) gives
\[ dZ = p \, dP + P \, dp, \]

whose integral is \( Z = Pp + C \), or
\[ \sqrt{pp + \cos^2 y} = \frac{pp}{\sqrt{pp + \cos^2 y}} + C; \]

which reduces to
\[ \cos^2 y = C \sqrt{pp + \cos^2 y}; \]

from which is deduced
\[ CC \, pp = \cos^2 y(\cos^2 y - CC), \]

or
\[ p = \frac{dy}{dx} = \frac{\cos y \sqrt{\cos^2 y - CC}}{C}. \]

Thus the relation between \( x \) and \( y \) is expressed in the separable differential equation
\[ dx = \frac{C \, dy}{\cos y \sqrt{\cos^2 y - CC}}. \]
and from this is obtained

\[ ds = dx \sqrt{pp + \cos^2 y} = \frac{dx \cos^2}{C}; \]

thus

\[ ds = \frac{dy \cos y}{\sqrt{\cos^2 y - CC}}. \]

and the arc itself

\[ s = \int \frac{dy \cos y}{\sqrt{\cos^2 y - CC}}. \]

**Corollary 1**

2. Thus, the equation

\[ dx = \frac{C \ dy}{\cos y \sqrt{\cos^2 y - CC}} \]

expresses the nature of the line AM, which has the property, that any portion whatever will be the shortest line that can be drawn between its endpoints on the surface of the sphere. Now, I have shown elsewhere that this line is also a great circle of the sphere; here it does not matter to our purpose, what relation this line has to the sphere, provided we know it is the shortest between its endpoints.

**Corollary 2**

3. Having found

\[ dx = \frac{C}{\cos y \sqrt{\cos^2 y - CC}} \]

we shall have

\[ Mn = dx \cos y = \frac{C \ dy}{\sqrt{\cos^2 y - CC}}. \]
Now $\frac{Mn}{mn}$ expresses the tangent of the angle $AMP$ and from this we have:

$$\tan AMP = \frac{C}{\sqrt{\cos^2 y - CC}}.$$ 

Furthermore, since

$$Mm = ds = \frac{dy \cos y}{\sqrt{\cos^2 y - CC}},$$ 

the fraction $\frac{Mn}{Mm}$ expresses the sine of the angle $AMP$, so that

$$\sin AMP = \frac{C}{\cos y} \quad \text{and} \quad \cos AMP = \frac{\sqrt{\cos^2 y - CC}}{\cos y}.$$ 

**COROLLARY 3**

4. Moreover, setting $y = 0$, the point $M$ comes to $A$ and then the fraction $\frac{dy}{dx}$ will express the tangent of the angle $PAM$, and $\frac{dy}{ds}$ its sine and $\frac{dx}{ds}$ its cosine. Now since $\cos y = 1$, we have

$$dx = \frac{C \frac{dy}{\sqrt{1 - CC}}} \quad \text{and} \quad ds = \frac{dy}{\sqrt{1 - CC}},$$

from which we conclude

$$\tan PAM = \frac{\sqrt{1 - CC}}{C}, \quad \sin PAM = \sqrt{1 - CC}, \quad \text{and} \quad \cos PAM = C.$$ 

**COROLLARY 4**

5. Thus, if we introduce this angle $PAM$ in place of the constant $C$, and set $PAM = \zeta$, we shall have, since $C = \cos \zeta$, the two following equations:

$$dx = \frac{dy \cos \zeta}{\cos y \sqrt{\cos^2 y - \cos^2 \zeta}} \quad \text{and} \quad ds = \frac{dy \cos y}{\sqrt{\cos^2 y - \cos^2 \zeta}}.$$ 

Moreover, if we name the angle $AMP = \theta$, we shall have

$$\tan \theta = \frac{\cos \zeta}{\sqrt{\cos^2 y - \cos^2 \zeta}}, \quad \sin \theta = \frac{\cos \zeta}{\cos y}, \quad \text{and} \quad \cos \theta = \frac{\sqrt{\cos^2 y - \cos^2 \zeta}}{\cos y}. $$
Corollary 5

6. It still remains to integrate the two differential equations which express the values of \(dx\) and \(ds\). By integration, it is found that

\[
x = \arcsin \frac{C \sin y}{\cos y \sqrt{1 - CC}}, \quad \text{or} \quad \sin x = \frac{C \sin y}{\cos y \sqrt{1 - CC}} = \frac{\cos \zeta \sin y}{\sin \zeta \cos y}
\]

and

\[
s = \arccos \frac{\sqrt{\cos^2 y - CC}}{\sqrt{1 - CC}}, \quad \text{or} \quad \cos s = \frac{\sqrt{\cos^2 y - CC}}{\sqrt{1 - CC}} = \frac{\sqrt{\cos^2 y - \cos^2 \zeta}}{\sin \zeta}
\]

Corollary 6

7. Here, then, are the quantities \(\zeta\) and \(y\), and the other quantities \(x, s, \theta\) determined from them:

\[
\sin x = \frac{\cos \zeta \sin y}{\sin \zeta \cos y} \quad \cos x = \frac{\sqrt{\cos^2 y - \cos^2 \zeta}}{\sin \zeta \cos y} \quad \tang x = \frac{\cos \zeta \sin y}{\sqrt{\cos^2 y - \cos^2 \zeta}}
\]

\[
\sin s = \frac{\sin y}{\sin \zeta} \quad \cos s = \frac{\sqrt{\cos^2 y - \cos^2 \zeta}}{\sin \zeta} \quad \tang s = \frac{\cos \zeta}{\sqrt{\cos^2 y - \cos^2 \zeta}}
\]

\[
\sin \theta = \frac{\cos \zeta}{\cos y} \quad \cos \theta = \frac{\sqrt{\cos^2 y - \cos^2 \zeta}}{\cos y} \quad \tang \theta = \frac{\cos \zeta}{\sqrt{\cos^2 y - \cos^2 \zeta}}
\]

Corollary 7

8. In the equations we have found, there is only one irrational formula,

\[
\sqrt{\cos^2 y - \cos \zeta};
\]
eliminating this, we obtain:

\[
\begin{align*}
\frac{\cos s}{\cos x} &= \cos y, & \frac{\cos \theta}{\cos x} &= \sin \zeta, & \frac{\cos \theta}{\cos s} &= \sin \zeta \\
\frac{\tan x}{\tan s} &= \cos \zeta, & \frac{\tan x}{\tan \theta} &= \sin y, & \frac{\tan s}{\tan \theta} &= \sin y \\
\sin x &= \frac{\cos \zeta \sin y}{\sin \zeta \cos y}, & \cos x \tan s &= \frac{\sin y}{\sin \zeta \cos y}, & \cos x \tan \theta &= \frac{\cos \zeta}{\sin \zeta \cos y}, \\
\cos s \tan x &= \frac{\cos \zeta \sin y}{\sin \zeta}, & \sin s &= \frac{\sin y}{\sin \zeta}, & \cos s \tan \theta &= \frac{\cos \zeta}{\sin \zeta}, \\
\cos \theta \tan x &= \frac{\cos \zeta \sin y}{\cos y}, & \cos \theta \tan s &= \frac{\sin y}{\cos y}, & \sin \theta &= \frac{\cos \zeta}{\cos y}.
\end{align*}
\]

**Corollary 8**

9. Having five quantities, \(x, y, s, \zeta,\) and \(\theta,\) which form part of the right spherical triangle \(APM,\) we take from the equalities above those which contain three of the quantities, and reduce them to a simpler form:

I. \(\cos s = \cos x \cos y,\) 
II. \(\cos \theta = \sin \zeta \cos x,\) 
III. \(\tan x = \cos \zeta \tan s,\) 
IV. \(\tan x = \sin y \tan \theta,\) 
V. \(\tan y = \sin x \tan \zeta;\)

whence, given any two quantities, one can find from them the three others, without the need to extract any roots, provided that one adds the tenth:

\(\text{X. } \sin x = \sin \theta \sin s.\)
Problem 2

10. Show the rules for the resolving of all cases of right spherical triangles.

Solution

Let the angles (Fig. 2) be marked \( A, B, C \), with \( C \) the right angle, and the sides by the lower-case letters \( a, b, c \), corresponding to their opposing angles, so that \( c \) is the hypotenuse and \( a \) and \( b \) the legs. Then comparing this triangle with the previous figure, we shall have

\[
\begin{align*}
 s &= c, & x &= b, & y &= a, & \zeta &= A, & \theta &= B.
\end{align*}
\]

The formulae above show that, given two quantities, the three others are determined; recalling these formulae will furnish the following resolutions for all possible cases:
The two given quantities | Determination of the three others
---|---
I. \(a, b\) & \(\cos c = \cos a \cdot \cos b,\) & \(\tang A = \frac{\tang a}{\sin b},\) & \(\tang B = \frac{\tang b}{\sin a}\)
II. \(a, c\) & \(\cos b = \frac{\cos a}{\cos c},\) & \(\sin A = \frac{\sin a}{\sin c},\) & \(\cos B = \frac{\tang c}{\sin b}\)
III. \(b, c\) & \(\cos a = \frac{\cos b}{\cos c},\) & \(\cos A = \frac{\tang b}{\sin c},\) & \(\sin B = \frac{\sin b}{\cos A}\)
IV. \(a, A\) & \(\sin b = \frac{\sin a}{\tang A},\) & \(\sin c = \frac{\sin a}{\sin A},\) & \(\sin B = \frac{\sin b}{\cos A}\)
V. \(a, B\) & \(\tang b = \sin a \tang B,\) & \(\tang c = \frac{\tang a}{\cos b},\) & \(\cos A = \cos a \sin B\)
VI. \(b, A\) & \(\tang a = \sin b \tang A,\) & \(\tang c = \frac{\cos A}{\tang b},\) & \(\cos B = \cos b \sin A\)
VII. \(b, B\) & \(\sin a = \frac{\tang b}{\tang B},\) & \(\sin c = \frac{\sin b}{\sin B},\) & \(\sin A = \frac{\cos B}{\cos b}\)
VIII. \(c, A\) & \(\sin a = \sin c \sin A,\) & \(\tang b = \tang c \cos A,\) & \(\tang B = \frac{1}{\cos c \tang A}\)
IX. \(c, B\) & \(\sin b = \sin c \sin B,\) & \(\tang a = \tang c \cos B,\) & \(\tang A = \frac{1}{\cos c \tang B}\)
X. \(A, B\) & \(\cos a = \frac{\cos A}{\sin B},\) & \(\cos b = \frac{\sin b}{\sin A},\) & \(\cos c = \frac{1}{\tang A \tang B}\)

**Corollary 1**

11. From the above it is evident that [the case of] side \(a\) with its opposing angle \(A\) enters into these formulae, exactly as does [the case of] side \(b\) with its opposing angle \(B\), so that it makes no difference which of the two sides \(a\) and \(b\) one wishes to take for the base, exactly as the nature of the subject requires.

**Corollary 2**

12. The large number of formulae which express the relationship between the various parts of the right triangle, are reduced to the following formulae, whose number is smaller; it suffices to learn these by heart.
I. \[ \sin c = \frac{\sin a}{\sin A} \quad \text{or} \quad \sin c = \frac{\sin b}{\sin B} \]

II. \[ \cos c = \cos a \cos b \]

III. \[ \cos c = \cot A \cot B \]

IV. \[ \cos A = \frac{\tan b}{\tan c} \quad \text{or} \quad \cos B = \frac{\tan a}{\tan c} \]

V. \[ \sin A = \frac{\cos b}{\cos B} \quad \text{or} \quad \sin B = \frac{\cos a}{\cos A} \]

VI. \[ \sin a = \frac{\tan b}{\tan B} \quad \text{or} \quad \sin b = \frac{\tan a}{\tan A} \]

13. It is only necessary to note these six formulae, which contain so many of the properties of right spherical triangles, and it will be possible to resolve every imaginable case of such triangles.